Reviewing Linear Systems

Read each worked example, then try the questions that follow it.

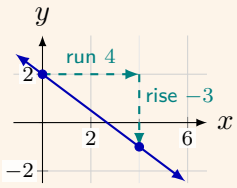
Graph Lines

Use the slope and the y-intercept

Graph the line $y = -\frac{3}{4}x + 2$.

The y-intercept is 2.
Start at $(0, 2)$.

The slope is $\frac{\text{rise}}{\text{run}} = \frac{-3}{4}$.
Run 4 to the right, then go down 3, to reach $(4, -1)$.



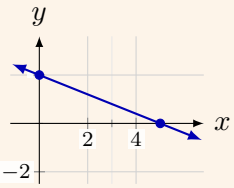
Use intercepts

Graph the line $2x + 5y = 10$.

At the x-intercept, $y = 0$.
$2x + 5(0) = 10$, so $x = 5$.

At the y-intercept, $x = 0$.
$2(0) + 5y = 10$, so $y = 2$.

Plot $(5, 0)$ and $(0, 2)$. Draw a line through them.

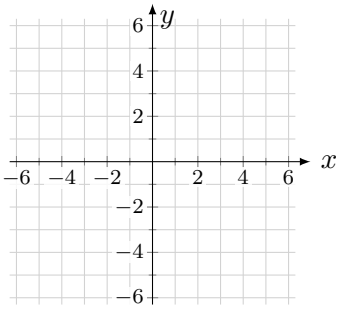


Which method is faster?

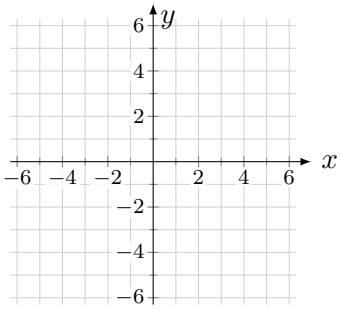
- If the equation is already in the form $y = mx + b$, use the slope and the y-intercept.
- If the equation is in the form $Ax + By = C$, use intercepts when A and B both divide evenly into C. For $2x + 5y = 10$: $10 \div 2 = 5$ and $10 \div 5 = 2$.
- If the equation is close to that form, like $Ax + By + C = 0$, move the constant to the other side first. Watch the sign: $3x + 2y + 12 = 0$ becomes $3x + 2y = -12$.
- If A does not divide evenly into C, the x-intercept is a fraction, which is hard to plot. If B still divides evenly into C, rearrange to the form $y = mx + b$ instead: $5x + 2y = 8$ becomes $y = -\frac{5}{2}x + 4$.

1. Graph each line. Choose the faster method.

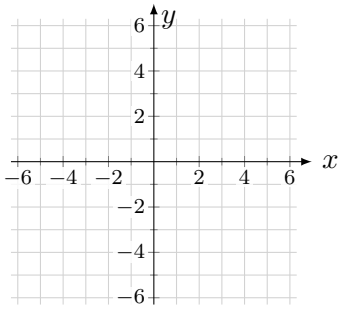
a) $y = 2x - 3$



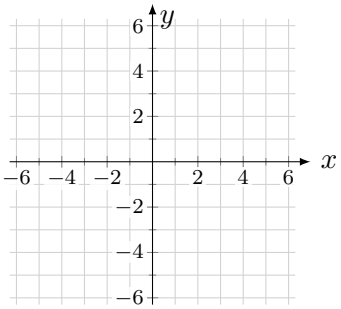
b) $y = -\frac{2}{3}x + 4$



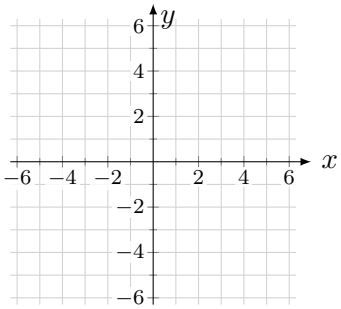
c) $3x + 2y = 6$



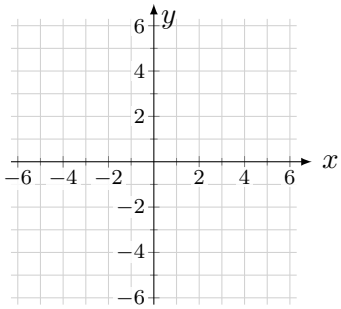
d) $5x - 2y = 10$



e) $4x + 3y + 12 = 0$



f) $3x - 4y = 8$

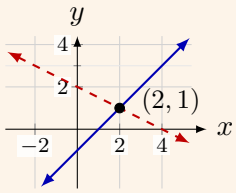


Solve Linear Systems by Graphing

Graph both lines on the same grid. A solution is a point that is on both lines. Two lines can meet in three ways. Compare the slopes and the y-intercepts to tell which.

Intersecting lines

$y = x - 1$
$y = -\frac{1}{2}x + 2$



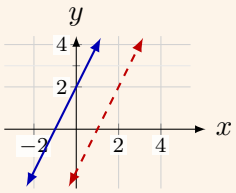
Slopes: different
y-intercepts: same or different

One solution: $(2, 1)$

The lines cross at one point.

Parallel lines

$y = 2x + 2$
$y = 2x - 2$



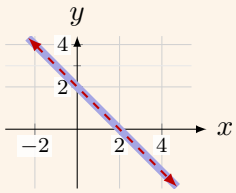
Slopes: same
y-intercepts: different

No solution

The lines never cross.

Parallel and coincident lines

$y = -x + 2$
$2x + 2y = 4$



Slopes: same
y-intercepts: same

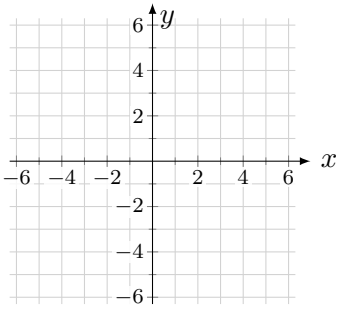
Infinitely many solutions

Both equations give the same line.

Rearrange to the form $y = mx + b$ before you compare: $2x + 2y = 4$ becomes $y = -x + 2$.

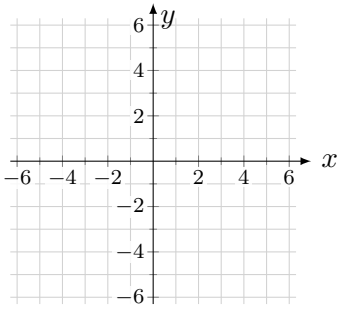
2. Solve each linear system by graphing. If there is no solution, or infinitely many solutions, say so. If the lines cross between grid points, estimate the solution.

a) $y = x + 1$
$y = -x + 3$



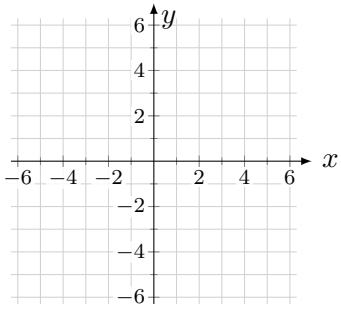
Solution: _____

b) $y = \frac{1}{2}x + 1$
$y = -x + 4$



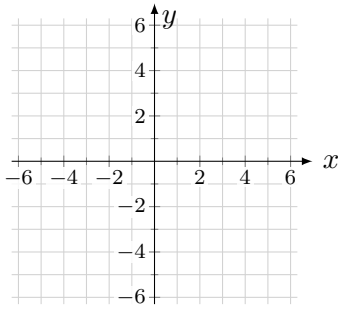
Solution: _____

c) $y = -\frac{1}{2}x + 3$
$x + 2y = -2$



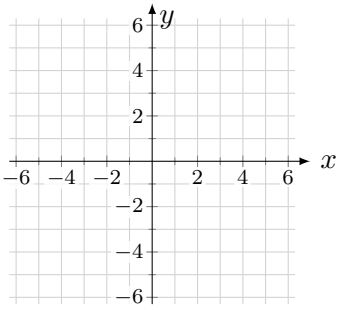
Solution: _____

d) $y = \frac{2}{3}x - 2$
$2x - 3y = 6$



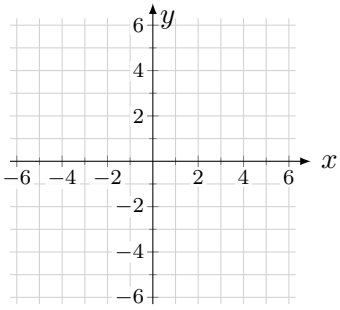
Solution: _____

e) $y = x + 2$
$y = -2x + 4$



Solution: _____

f) $2x - y = 1$
$x + 2y = 6$



Solution: _____

Solve Linear Systems Algebraically

Organize every solution. Label the equations (A) and (B). Write a short sentence to explain each stage.

Substitution

$$y = 2x - 5 \quad \text{(A)}$$
$$3x + 2y = 4 \quad \text{(B)}$$

(A) is already solved for y. Substitute (A) into (B).

$$3x + 2(2x - 5) = 4$$
$$3x + 4x - 10 = 4$$
$$7x = 14$$
$$x = 2$$

Substitute $x = 2$ into (A) to find y.

$$y = 2(2) - 5$$
$$y = -1$$

The solution is $(2, -1)$.

Elimination

$$3x + 2y = 7 \quad \text{(A)}$$
$$5x - 4y = 19 \quad \text{(B)}$$

Multiply (A) by 2, then add to eliminate y.

$$6x + 4y = 14 \quad \text{(C)} \quad 2\times \text{(A)}$$
$$+\ 5x - 4y = 19 \quad \text{(B)}$$
$$11x = 33$$
$$x = 3$$

Substitute $x = 3$ into (A) to find y.

$$3(3) + 2y = 7$$
$$2y = -2, \text{ so } y = -1$$

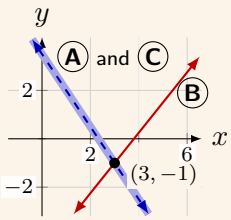
The solution is $(3, -1)$.

Why does elimination work? (A) and (C) are *equivalent equations*. The coefficients differ, but the rule that connects the input, x, to the output, y, is identical:

(A) $3x + 2y = 7$ rearranges to $y = -\frac{3}{2}x + \frac{7}{2}$

(C) $6x + 4y = 14$ rearranges to $y = -\frac{3}{2}x + \frac{7}{2}$

So (A) and (C) have the same graph. The solution does not change.



Use substitution when one equation is already in the form $y = mx + b$, or when a variable has a coefficient of 1, like the x in $x + 3y = 9$. That variable is easy to isolate.

Use elimination when both equations are in the form $Ax + By = C$ or $Ax + By + C = 0$ and no variable has a coefficient of 1 or -1. Isolating a variable in $3x + 2y = 7$ would create fractions.

3. Solve each system on a separate sheet of lined paper. Use your judgement to choose the best method.

a) $y = -2x + 3$, $y = x - 6$

b) $y = 3x - 1$, $2x + y = 9$

c) $3x + 4y = 10$, $5x - 4y = 6$

d) $x + 2y = 7$, $3x - y = 7$

e) $2x - 5y = 6$, $4x - 10y = 5$

f) $3x + 2y - 4 = 0$, $5x + 3y - 7 = 0$

g) $2x + y = 5$, $6x + 3y = 15$

h) $2x + 4y = 5$, $6x - 4y = -1$

No Solution, or Infinitely Many Solutions?

If both variables are eliminated at once, the statement that is left tells you the kind of system.

$$x + 2y = 5 \quad \text{(A)}$$
$$3x + 6y = 9 \quad \text{(B)}$$

Multiply (A) by 3, then subtract (B).

$$3x + 6y = 15 \quad \text{(C)} \quad 3\times \text{(A)}$$
$$-\ 3x + 6y = 9 \quad \text{(B)}$$
$$0x + 0y = 6$$

The statement $0 = 6$ is never true.
Parallel lines: **no solution**.

$$y = 3x - 2 \quad \text{(A)}$$
$$6x - 2y = 4 \quad \text{(B)}$$

Substitute (A) into (B).

$$6x - 2(3x - 2) = 4$$
$$6x - 6x + 4 = 4$$
$$0x = 0$$

The statement $0x = 0$ is true for every value of x.
Coincident lines: **infinitely many solutions**.

Model Problems with Linear Systems

From words to algebra

First say what each variable means. Then translate each phrase.

x is the larger of two numbers, and y is the smaller.

the sum of the numbers	$x + y$
the smaller subtracted from the larger	$x - y$
twice the larger, minus the smaller	$2x - y$
The larger *is* 5 more than the smaller	$x = y + 5$

x adult tickets at \$12 and y child tickets at \$8 are sold.

the number of tickets sold	$x + y$
the revenue, in dollars	$12x + 8y$

A jar holds x dimes and y quarters.

the value of the coins, in cents	$10x + 25y$

A gym charges \$$x$ to join, plus \$$y$ per month.

the cost of 6 months	$x + 6y$

\$$x$ is invested at 4%/year, and \$$y$ at 5%/year.

the total invested	$x + y$
the interest earned in a year	$0.04x + 0.05y$

A boat travels at x km/h in still water. The current is y km/h.

the speed with the current	$x + y$
the speed against the current	$x - y$

An expression becomes an equation when the problem gives its value. The word *is* often becomes $=$.

Define each variable with a complete sentence

A rectangular garden has a perimeter of 24 m. Its length is 200 cm more than its width. Find the length and the width of the garden.

Working in metres:

Let x be the length of the garden, in metres.
Let y be the width of the garden, in metres.

$2x + 2y = 24$ ← the perimeter
$x = y + 2$ ← length compared with width

Working in centimetres:

Let x be the length of the garden, in centimetres.
Let y be the width of the garden, in centimetres.

$2x + 2y = 2400$ ← the perimeter
$x = y + 200$ ← length compared with width

The equations differ, but both set-ups are correct, because each sentence states the unit. "Let x be the length" is not enough: a reader cannot tell whether $x = 7$ or $x = 700$. Use one unit throughout, and note beside each equation what it models.

4. On lined paper, **set up** each problem, but do **not** solve it. Use complete sentences to define what x and y represent, with units where they apply. Write two equations. Beside each equation, note what it models.

a) A food truck sold 85 tacos and burritos during one lunch hour. A taco costs \$4 and a burrito costs \$9. Sales totalled \$515. How many tacos and how many burritos were sold?

b) A climbing gym charges a membership fee, plus a fee for each visit. Aiko made 8 visits and paid \$106 in all. Ben made 15 visits and paid \$155 in all. Find the membership fee and the fee for one visit.

c) Dev invested \$8000. He invested part of it at 3%/year and the rest at 5%/year. After one year, he had earned \$340 in interest. How much did Dev invest at each rate?

d) Paddling with the current, a canoeist travels at 9 km/h. Paddling against the current, she travels at 3 km/h. Find her paddling speed in still water, and the speed of the current.

e) A carpenter cuts a board that is 3 m long into two pieces. One piece is 40 cm longer than the other. How long is each piece?

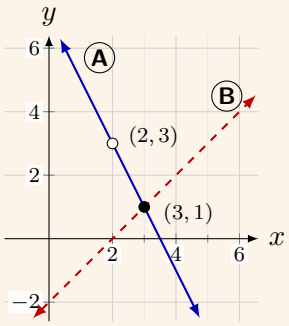
f) Sam has 40 coins. All of them are dimes or quarters. Together, the coins are worth \$6.25. How many dimes and how many quarters does Sam have?

Check a Solution: Left Side and Right Side

The solution to a linear system is the point of intersection: a point that is on *both* lines. One algebra error while solving can produce a point that is on only one of the lines, or on neither. To be sure, substitute the point into both original equations. If the left side (LS) equals the right side (RS), the point is on that line.

For the system below, a friend found the solution $(2, 3)$. Is the friend correct?

$2x + y = 7$ (A) $\qquad x - y = 2$ (B)



○ $(2, 3)$ is on (A) only
● $(3, 1)$ is on both lines

Check $(2, 3)$ in (A)

$$\begin{aligned} \text{LS} &= 2x + y \\ &= 2(2) + 3 \\ &= 4 + 3 \\ &= 7 \end{aligned}$$

RS $= 7$

LS = RS ✓ on line (A)

Check $(2, 3)$ in (B)

$$\begin{aligned} \text{LS} &= x - y \\ &= 2 - 3 \\ &= -1 \end{aligned}$$

RS $= 2$

LS ≠ RS not on line (B)

So $(2, 3)$ is on (A) only. It is **not** the solution. The graph agrees.

Another way to set out a check: a T-chart

The correct solution is $(3, 1)$. In a T-chart, each side has its own column.

Check $(3, 1)$ in (A)

LS	RS
$2x + y$	7
$= 2(3) + 1$	
$= 6 + 1$	
$= 7$	

LS = RS ✓

Check $(3, 1)$ in (B)

LS	RS
$x - y$	2
$= 3 - 1$	
$= 2$	

LS = RS ✓

Use the equals sign with care.

Begin a line with = only when that line is equal to the line above it. Do not write = between the two sides until the check is finished: the check is what shows whether they are equal.

LS = RS in both, so $(3, 1)$ is on both lines.

What a check tells you

LS = RS in both equations	the point is on both lines	it is a solution
LS = RS in only one equation	the point is on one line only	it is not a solution
LS ≠ RS in both equations	the point is on neither line	it is not a solution

If the lines are coincident, a point that checks in both equations is just one of infinitely many solutions. A check cannot show this. Look at the equations: is one a multiple of the other?

5. A friend solved each system and found the point shown. On lined paper, complete a LS/RS check in both equations. Then say where the point is: on both lines, on one line only (which one?), or on neither line. Is it the solution, one of many solutions, or not a solution?

a) $4x + y = 5$ (A)
$2x - 3y = 13$ (B)
Friend's solution: $(1, 1)$

b) $3x + 2y = 12$ (A)
$x - y = -1$ (B)
Friend's solution: $(2, 3)$

c) $5x - 2y = 4$ (A)
$x + 3y = 11$ (B)
Friend's solution: $(5, 2)$

d) $x + y = 6$ (A)
$x - 2y = 3$ (B)
Friend's solution: $(4, 1)$

e) $2x - y = 3$ (A)
$6x - 3y = 9$ (B)
Friend's solution: $(2, 1)$

f) $3x + 4y = 1$ (A)
$2x - y = -3$ (B)
Friend's solution: $(-1, 1)$

Mixed Practice: Knowing Where to Start

On a test, nothing tells you which page of this review a question comes from. When you are not sure how to begin:

- Read the whole question. What is it asking for: a solution, a set-up, a check, or an explanation?
- Look at the form of the equations before you choose a method.
- Write down what you know. Label the equations, define the variables, or make a quick sketch.
- If both variables disappear, ask what the statement that is left means before you decide it is a mistake.

These questions are in mixed order. Decide for yourself how to approach each one. Work on lined paper.

6. Solve the linear system $y = 2x - 4$ and $4x - 3y = 10$. Name the method you chose, and say why.

7. A store sold 60 jackets. Some sold at the regular price of \$80, and the rest at the sale price of \$50. Sales totalled \$4050. Set up a linear system to find how many jackets sold at each price. Do not solve.

8. While solving a linear system, Jo reaches the statement $0 = 5$ and says, "I must have made a mistake." Is Jo necessarily right? Explain.

9. The point $(2, -1)$ is the solution to the linear system $mx + 3y = 5$ and $4x + ny = 11$. Find the values of m and n.

10. Solve the linear system $4x + 3y = 6$ and $5x - 2y = 19$. Name the method you chose, and say why.

11. Flying with the wind, a plane travels 1800 km in 3 hours. The return trip, against the same wind, takes 3.6 hours. Set up a linear system to find the speed of the plane in still air and the speed of the wind. Do not solve.

12. Dan substituted $y = 3x - 2$ into $2x - y = 5$ and wrote: $2x - 3x - 2 = 5$, so $-x = 7$, so $x = -7$. Find Dan's error. Solve the system correctly. Then check your solution with LS/RS in both equations.

13. A classmate says, "Graphing always gives the exact solution to a linear system." Do you agree? Explain, and give an example of a system to support your answer.

14. For what value of k does the linear system $y = 2x + k$ and $6x - 3y = 12$ have infinitely many solutions? How many solutions are there for any other value of k? Explain.

15. Phone plan A costs \$20 per month, plus \$0.10 per minute of calls. Plan B costs \$5 per month, plus \$0.25 per minute. Set up a linear system that models the monthly cost of each plan. Do not solve. What would the solution to the system tell you?

16. Solve the linear system $x = 3y + 5$ and $2x - 6y = 10$. What does your result tell you about the two lines?

17. What does it mean for a point to be "a solution" to a linear system? Answer in two ways: in terms of the equations, and in terms of the graph.

18. One line of a linear system is $y = -3x + 2$. Write an equation for a second line so that the system has: a) no solutions; b) infinitely many solutions, using an equation that does not look the same as the first; c) exactly one solution, at $(1, -1)$.

19. A rectangular field is enclosed by 1.2 km of fencing. The field is 30 m longer than it is wide. Set up a linear system to find the length and the width of the field. Do not solve.

Reviewing Linear Systems — Final Answers

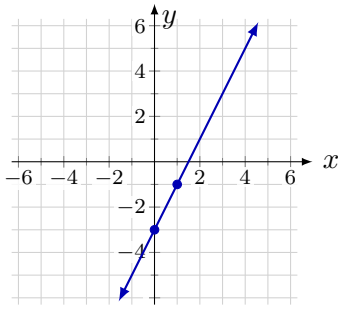
Final answers only.

Graph lines

1. The points used to draw each line are marked.

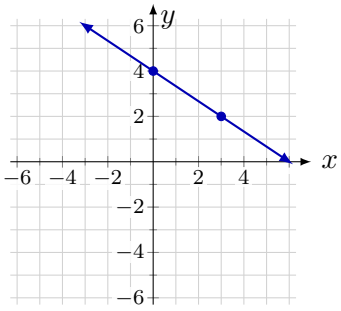
a) $y = 2x - 3$

slope 2, y-intercept -3



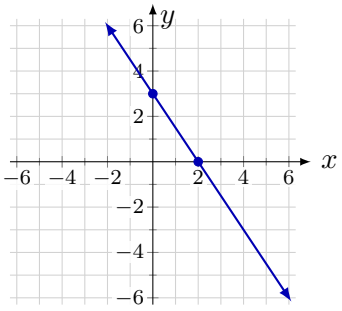
b) $y = -\frac{2}{3}x + 4$

slope $-\frac{2}{3}$, y-intercept 4



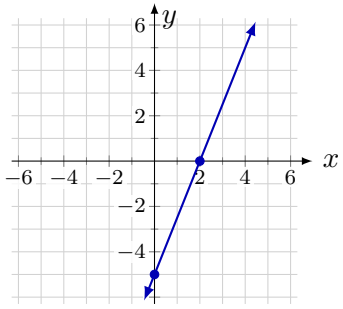
c) $3x + 2y = 6$

x-intercept 2, y-intercept 3



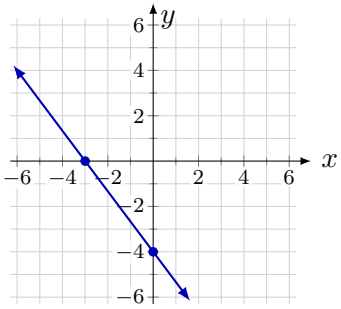
d) $5x - 2y = 10$

x-intercept 2, y-intercept -5



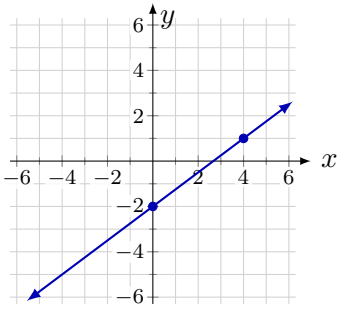
e) $4x + 3y + 12 = 0$

x-intercept -3, y-intercept -4



f) $3x - 4y = 8$

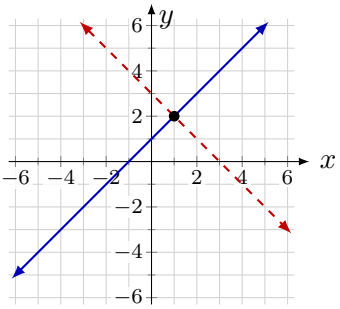
slope $\frac{3}{4}$, y-intercept -2



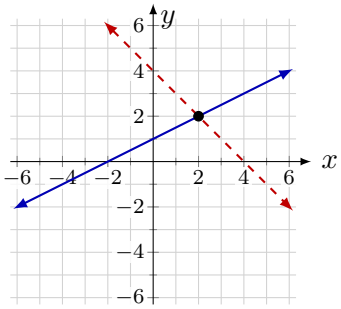
Solve linear systems by graphing

2. For e) and f), the lines cross between grid points. Accept any close estimate.

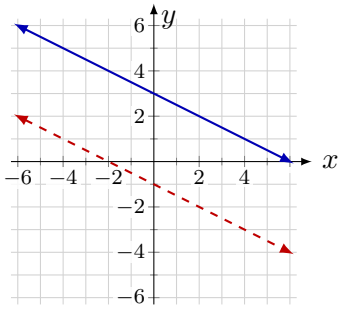
a) $y = x + 1$
$y = -x + 3$
$(1, 2)$



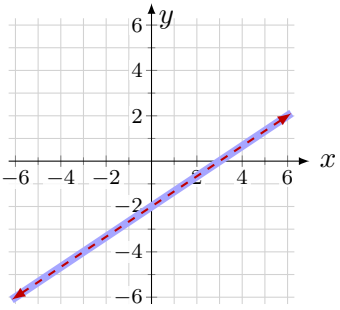
b) $y = \frac{1}{2}x + 1$
$y = -x + 4$
$(2, 2)$



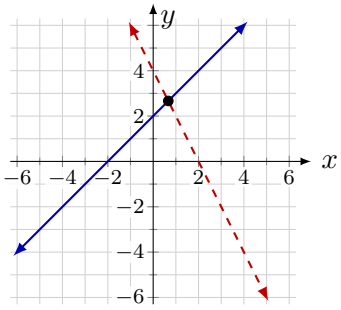
c) $y = -\frac{1}{2}x + 3$
$x + 2y = -2$
no solution



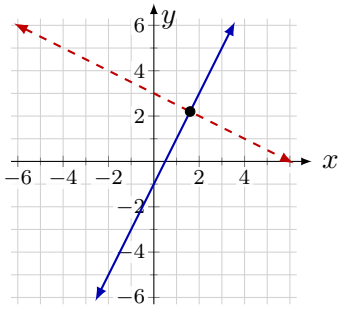
d) $y = \frac{2}{3}x - 2$
$2x - 3y = 6$
infinitely many solutions



e) $y = x + 2$
$y = -2x + 4$
about $(0.7, 2.7)$; exactly $\left(\frac{2}{3}, \frac{8}{3}\right)$



f) $2x - y = 1$
$x + 2y = 6$
$(1.6, 2.2)$, or $\left(\frac{8}{5}, \frac{11}{5}\right)$



Solve linear systems algebraically

3. The method named is the more efficient one. Systems e) and g) are the special cases.

a) $(3, -3)$
substitution

b) $(2, 5)$
substitution

c) $(2, 1)$
elimination

d) $(3, 2)$
either method

e) no solution
$0x + 0y = 7$

f) $(2, -1)$
elimination

g) infinitely many
$0x = 0$

h) $(\frac{1}{2}, 1)$
elimination

Model problems with linear systems

4. Other choices of variables and units are possible. Solutions are given for reference; students were not asked to solve.

a) Let x be the number of tacos sold. Let y be the number of burritos sold.
$x + y = 85$ (items sold) $4x + 9y = 515$ (sales, in dollars)
Solution: 50 tacos, 35 burritos

b) Let x be the membership fee, in dollars. Let y be the fee for one visit, in dollars.
$x + 8y = 106$ (what Aiko paid) $x + 15y = 155$ (what Ben paid)
Solution: membership \$50, each visit \$7

c) Let x be the amount invested at 3%, in dollars. Let y be the amount invested at 5%, in dollars.
$x + y = 8000$ (money invested) $0.03x + 0.05y = 340$ (interest earned)
Solution: \$3000 at 3%, \$5000 at 5%

d) Let x be the paddling speed in still water, in km/h. Let y be the speed of the current, in km/h.
$x + y = 9$ (speed with the current) $x - y = 3$ (speed against the current)
Solution: paddling 6 km/h, current 3 km/h

e) Let x be the length of the longer piece, in centimetres. Let y be the length of the shorter piece, in centimetres.
$x + y = 300$ (whole board) $x = y + 40$ (lengths compared)
Solution: 170 cm and 130 cm
Or, with x and y in metres: $x + y = 3$ and $x = y + 0.4$ (1.7 m and 1.3 m). Mixing the units, as in $x + y = 3$ with $x = y + 40$, is the error to watch for.

f) Let x be the number of dimes. Let y be the number of quarters.
$x + y = 40$ (number of coins) $10x + 25y = 625$ (value of the coins, in cents)
Solution: 25 dimes, 15 quarters
Or, with the value in dollars: $0.10x + 0.25y = 6.25$.

Check a solution

5. The value of LS is given for each check, with RS in brackets where they differ.

a) (A): LS = 5 = RS
(B): LS = −1 (RS = 13)
On line (A) only:
not a solution.

b) (A): LS = 12 = RS
(B): LS = −1 = RS
On both lines:
the solution.

c) (A): LS = 21 (RS = 4)
(B): LS = 11 = RS
On line (B) only:
not a solution.

d) (A): LS = 5 (RS = 6)
(B): LS = 2 (RS = 3)
On neither line:
not a solution.

e) (A): LS = 3 = RS
(B): LS = 9 = RS
On both lines, but (B) is 3× (A): the lines are coincident, so it is one of infinitely many solutions.

f) (A): LS = 1 = RS
(B): LS = −3 = RS
On both lines:
the solution.

Mixed practice

For set-up questions, other variables and units are possible. Students were not asked to solve.

6. $(1, -2)$. Substitution: the first equation is already solved for y.

7. Let x be the number of jackets sold at the regular price. Let y be the number sold at the sale price.
$x + y = 60$ (jackets sold) $\quad 80x + 50y = 4050$ (sales, in dollars)
Solution: 35 at the regular price, 25 at the sale price

8. No. If both variables are eliminated and the statement that is left is false, the lines are parallel and the system has no solutions. Jo should recheck the algebra, but $0 = 5$ can be a correct result.

9. $m = 4$ and $n = -3$ (substitute $x = 2$, $y = -1$: $2m - 3 = 5$ and $8 - n = 11$)

10. $(3, -2)$. Elimination: both equations are in the form $Ax + By = C$, and no variable has a coefficient of 1.

11. Let x be the speed of the plane in still air, in km/h. Let y be the speed of the wind, in km/h.
$3(x + y) = 1800$ (distance with the wind) $\quad 3.6(x - y) = 1800$ (distance against the wind)
Solution: plane 550 km/h, wind 50 km/h
Or, dividing each distance by its time: $x + y = 600$ and $x - y = 500$.

12. Dan did not subtract the whole expression: $2x - (3x - 2) = 2x - 3x + 2$. Correct work: $-x + 2 = 5$, so $x = -3$ and $y = 3(-3) - 2 = -11$. The solution is $(-3, -11)$.
Check: $3(-3) - 2 = -11$ and $2(-3) - (-11) = 5$, so LS = RS in both equations.

13. No. A graph gives the exact solution only when the lines cross at a point that can be read exactly, such as a grid point. For example, $y = x + 2$ and $y = -2x + 4$ cross at $\left(\frac{2}{3}, \frac{8}{3}\right)$, which a graph can only estimate. Substitution or elimination gives the exact values.

14. $k = -4$, because $6x - 3y = 12$ rearranges to $y = 2x - 4$, so the lines are coincident. For any other k, the lines have the same slope but different y-intercepts: they are parallel, and there are no solutions.

15. Let x be the number of minutes of calls in a month. Let y be the cost for the month, in dollars.
$y = 20 + 0.10x$ (Plan A) $\quad y = 5 + 0.25x$ (Plan B)
Solution: 100 minutes, \$30
The solution is the number of minutes for which both plans cost the same, and that cost.

16. Substituting gives $6y + 10 - 6y = 10$, or $10 = 10$, which is always true. There are infinitely many solutions: the lines are coincident.

17. Equations: its x- and y-values make both equations true (LS = RS in both). Graph: the point is on both lines, so it is where the lines cross.

18. Answers vary. a) same slope, different y-intercept, e.g. $y = -3x + 5$. b) an equivalent equation, e.g. $6x + 2y = 4$. c) any other line through $(1, -1)$, e.g. $y = x - 2$.

19. Let x be the length of the field, in metres. Let y be the width of the field, in metres.
$2x + 2y = 1200$ (the fencing, in metres) $\quad x = y + 30$ (length compared with width)
Solution: 315 m by 285 m
Or, with x and y in kilometres: $2x + 2y = 1.2$ and $x = y + 0.03$.