Reviewing Quadratic Relations: Days 1 to 5

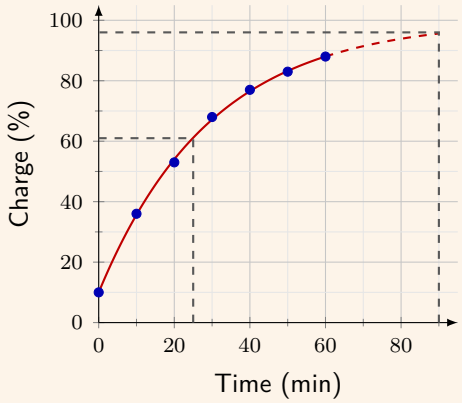
Adapted from Principles of Mathematics 10, McGraw-Hill Ryerson, 2007.

Five topics from Days 1 to 5. Read each worked example, then try the questions after it. Plot and sketch on the grids printed here; work the rest on lined paper. The last section mixes all five topics.

Relations and Models

I charged a phone and wrote down its charge every 10 minutes. I controlled the times, so **time is the independent variable**: it comes first in each ordered pair and goes on the horizontal axis. The charge depends on the time: it is the **dependent variable**, on the vertical axis.

Time (min)	Charge (%)
0	10
10	36
20	53
30	68
40	77
50	83
60	88



Describe the scatter plot in words

- **Direction:** positive. As time increases, the charge increases.
- **Strength:** strong. The points lie close to a smooth curve.
- **Shape:** non-linear. It rises quickly, then more slowly.

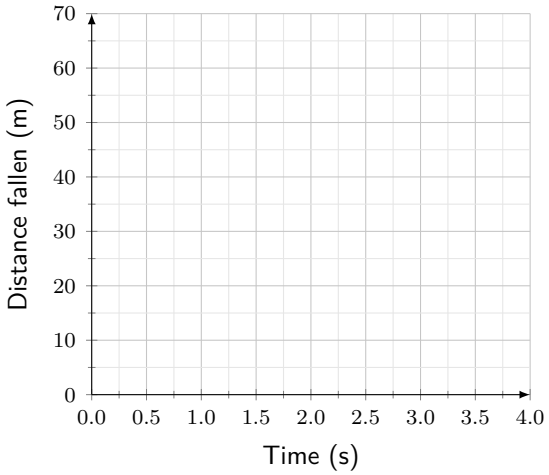
The points bend, so I drew a **curve of best fit**, with about as many points above it as below.

Predict from the curve. At 25 min the curve gives about 61%. Twenty-five minutes lies inside the data (0 to 60 min), so this is an **interpolation**: a prediction between known values. At 90 min the curve gives about 96%. Ninety minutes lies beyond the data, so this is an **extrapolation**: a prediction outside the known values. I trust the interpolation more. An extrapolation assumes the trend carries on, and the charge can never pass 100%.

When a model is not justified. If the points form a wide cloud with no clear trend, a line or curve through them predicts almost nothing. Check for a trend before you draw one.

1. A stone is dropped into a deep well. A video shows how far the stone has fallen every half second.

Time (s)	Fallen (m)
0.0	0.0
0.5	1.3
1.0	4.8
1.5	11.1
2.0	19.7
2.5	30.4
3.0	44.3



a) Which variable is independent, and which is dependent?

b) Plot the data on the grid. Draw a curve of best fit, and extend it to 3.5 s.

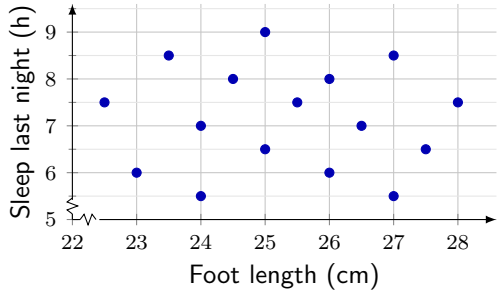
c) Describe the relation in words: its direction, strength, and shape.

d) Use your curve to estimate the distance fallen at 1.75 s and at 3.5 s. Say which estimate is an interpolation and which is an extrapolation.

e) The stone hit the water 3.2 s after it was dropped. Which estimate do you trust more? Why?

2. Work on lined paper.

a) A café writes down the temperature at noon and the number of cold drinks it sells each day. Which variable is independent? Explain.



b) The scatter plot shows the foot length of 16 students and the hours each one slept last night. Describe the relation in words.

c) Mei wants to draw a line of best fit to predict a student's sleep from foot length. Is a model justified? Explain.

Quadratic Relations and Finite Differences

With evenly spaced x values, the **first differences** are the differences of consecutive y values. The **second differences** are the differences of consecutive first differences.

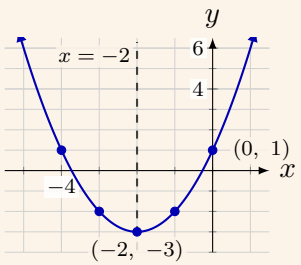
Linear: the first differences are constant.
Quadratic: the first differences are not constant, but the second differences are.
Neither: neither column is constant.

The rule needs **evenly spaced** x values. Measured data give second differences that are only *approximately* constant.

$y = x^2 + 4x + 1$

x	y	First differences	Second differences
-4	1		
		$-2 - 1 = -3$	
-3	-2		$-1 - (-3) = 2$
		$-3 - (-2) = -1$	
-2	-3		$1 - (-1) = 2$
		$-2 - (-3) = 1$	
-1	-2		$3 - 1 = 2$
		$1 - (-2) = 3$	
0	1		

Each difference is the lower value minus the one above it.



The second differences are all 2, so $y = x^2 + 4x + 1$ is **quadratic**. Its graph is a U-shaped curve, a **parabola**.

- It **opens up**. Its turning point, the **vertex**, is $(-2,\ -3)$.
- It is symmetric about the vertical line through the vertex, the **axis of symmetry**, $x = -2$.
- It opens up, so -3 is its **minimum value**.
- The y**-intercept** is the value of y when $x = 0$. Every term with an x becomes 0, leaving the constant term: $y = 0 + 0 + 1 = 1$.

3. Complete the difference columns for each table. Then decide whether the relation is linear, quadratic, or neither, and say how you know.

a)

x	y	First differences	Second differences
0	1		
1	4	$4-1=3$	
2	5	$5-4=1$	$1-3=-2$
3	4		
4	1		

Each difference is the lower value minus the one above it.

b)

x	y	First differences	Second differences
-2	7		
-1	4		
0	1		
1	-2		
2	-5		

c)

x	y	First differences	Second differences
0	1		
1	3		
2	7		
3	15		
4	31		

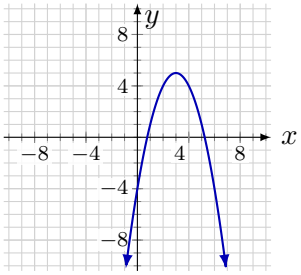
4. Ana plotted the points in this table. "The points curve upward," she said, "so the relation must be quadratic."

x	y	First differences	Second differences
0	3		
1	4		
2	6		
3	10		
4	18		

a) Complete the difference columns.

b) What is wrong with Ana's reasoning? Is the relation linear, quadratic, or neither?

5. You have this relation as a graph and as an equation: $y = -x^2 + 6x - 4$.



a) Describe its key features.

Opens _____ Max or min value _____

Vertex _____ y-intercept _____

Axis of symmetry _____

b) Show how the equation gives the y-intercept, with no graph.

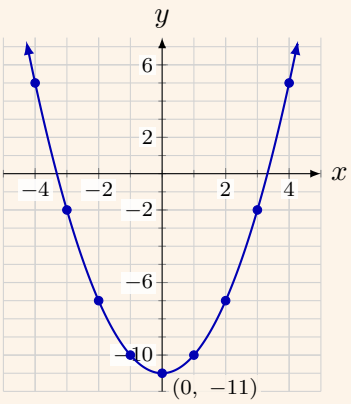
The Parent Parabola and Vertical Translations: $y = x^2 + k$

x	y
−3	9
−2	4
−1	1
0	0
1	1
2	4
3	9

$y = x^2$

The **parent parabola** $y = x^2$ opens upward. Its **vertex** is $(0,\ 0)$, so its **minimum value** is 0. Its **axis of symmetry** is the line $x = 0$. Its second differences are all 2.

Adding k to every y-value slides the whole graph k units: **up** when $k > 0$, **down** when $k < 0$. This is a **vertical translation**. The vertex is $(0,\ k)$, and the y-intercept and the minimum value are both k. The shape, the direction of opening, the axis of symmetry, and the differences do not change.

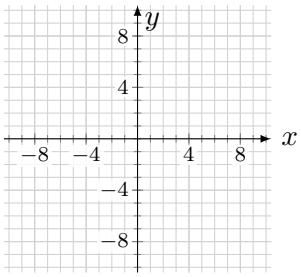


Sketch $y = x^2 - 11$ without a table. Plot the vertex $(0,\ -11)$. Step out 1 across and 1 up, 2 across and 4 up, 3 across and 9 up, 4 across and 16 up, on each side: $(\pm 1,\ -10)$, $(\pm 2,\ -7)$, $(\pm 3,\ -2)$, $(\pm 4,\ 5)$.

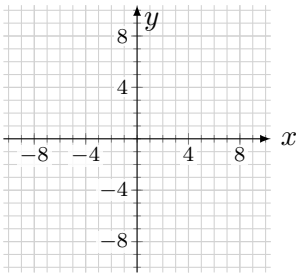
Where it crosses. When $k < 0$ the parabola crosses the x-axis twice; when $k = 0$ it touches it at the origin; when $k > 0$ it never meets it. Here $3^2 - 11 = -2$ is below the axis and $4^2 - 11 = 5$ is above it, so one crossing lies between 3 and 4. By symmetry, the other lies between -4 and -3.

6. Sketch each parabola without a table. Label the vertex.

a) $y = x^2 - 6.5$



b) $y = x^2 + 2.5$



7. Each row describes a parabola of the form $y = x^2 + k$. Complete the table.

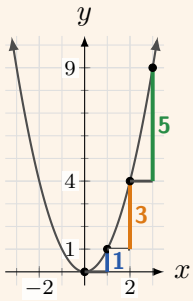
	Equation	In words	Vertex	Times it crosses the x-axis
a)	$y = x^2 + 9$			
b)		$y = x^2$ moved 14 units down		
c)			$(0,\ -3.5)$	
d)				1: it only touches the x-axis

8. The parabola $y = x^2 - 14$ crosses the x-axis twice. Between which two whole numbers does each crossing lie? Explain how you know. Work on lined paper.

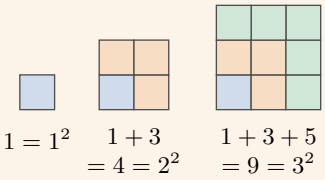
Stretches, Compressions, Reflections, and the Step Pattern

In $y = ax^2$, every y-value of $y = x^2$ is multiplied by a. **Vertical stretch** ($a > 1$ or $a < -1$): the graph is narrower. **Vertical compression** (a between -1 and 1, not 0): the graph is wider. **Reflection in the x-axis** ($a < 0$): the graph opens down, and the vertex is a maximum. The **size** of a sets the width; the **sign** of a sets the direction.

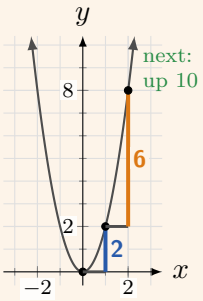
The staircase on $y = x^2$



The heights are squares



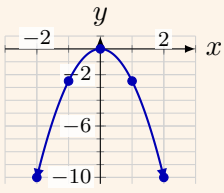
The staircase on $y = 2x^2$



Why 1, 3, 5? From the vertex of $y = x^2$, the heights are $1^2 = 1$, $2^2 = 4$, and $3^2 = 9$. Picture each height as a square of unit squares. Each square grows from the one before by an L-shaped layer: 3 squares, then 5. Each new L has one more square on each arm, so it holds 2 more than the L before it. Each L is one rise of the staircase. $y = 2x^2$ doubles every height, so it doubles every rise: 2, 6, 10. For $y = ax^2$, the steps are $1a$, $3a$, $5a$, and when $a < 0$ each step goes down.

Sketch by Steps: $y = -2.5x^2$

Plot the vertex $(0,\ 0)$. Here $a = -2.5$: it is a vertical stretch by a factor of 2.5 and a reflection in the x-axis. The steps go down 2.5, then $3 \times 2.5 = 7.5$. Go across 1 and down 2.5 to $(1,\ -2.5)$, then across 1 and down 7.5 to $(2,\ -10)$. The next step, down 12.5, would leave a -10 to 10 grid, so stop. Mirror the points in the axis $x = 0$, and join them with a smooth curve.

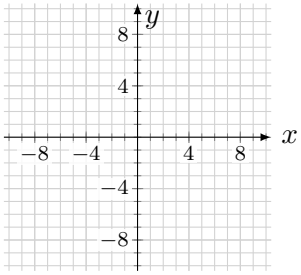


9. For each relation, say in words what a does to the graph of $y = x^2$. Then give the first three steps from the vertex. Work on lined paper.

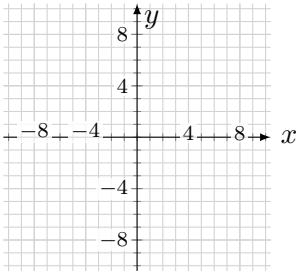
a) $y = 7x^2$ b) $y = -\frac{1}{5}x^2$ c) $y = -6x^2$ d) $y = 0.3x^2$

10. Sketch each parabola by the step pattern. Label the vertex and list the steps you used.

a) $y = 0.6x^2$



b) $y = -\frac{2}{3}x^2$



11. Work on lined paper.

a) Use the step pattern to give the first four steps from the vertex of $y = 0.7x^2$.

b) Add your steps to find y when $x = 4$. Then check by substituting $x = 4$.

c) Your steps are the first differences of a table that starts at the vertex. What are its second differences?

Horizontal Translations and Vertex Form: $y = a(x - h)^2 + k$

$y = (x-h)^2$ moves $y = x^2$ **right** h units when $h > 0$ and **left** when $h < 0$: a **horizontal translation**. The vertex is where the bracket is zero. **Watch the sign of h:** $y = (x + 5)^2$ is $y = (x - (-5))^2$, so $h = -5$, 5 units *left*.

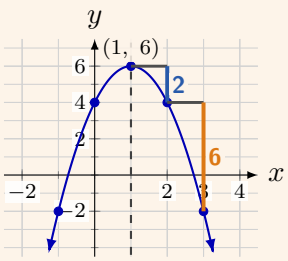
Put them together: stretch or compress by the size of a (reflect if $a < 0$), then translate h across and k up or down.

Vertex	$(h,\ k)$
Axis of symmetry	$x = h$
Opens	up when $a > 0$; down when $a < 0$
Max or min	minimum k when $a > 0$; maximum k when $a < 0$

From the vertex, go 1 right and 2 down to $(2,\ 4)$, then 1 right and 6 down to $(3,\ -2)$. The next step would reach $(4,\ -12)$, off the grid, so stop. Mirror the points in the axis $x = 1$: $(0,\ 4)$ and $(-1,\ -2)$. Join them with a smooth curve. Check one point: at $x = 3$, $-2(3-1)^2+6 = -8+6 = -2$. The axis of symmetry is $x = 1$. It opens down, so 6 is its **maximum value**.

Sketch $y = -2(x - 1)^2 + 6$.

In words: $y = x^2$ stretched by a factor of 2, reflected, then moved 1 right and 6 up. Plot the vertex $(1,\ 6)$. Here $a = -2$, so the steps go down 2, 6, 10.

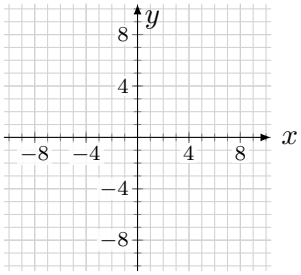


12. Complete the table. Do not make a table of values or a graph.

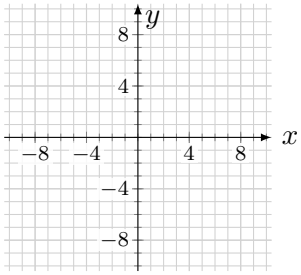
	Equation	Vertex	Axis of symmetry	Opens	Max or min value	Stretch, compression, or reflection
a)	$y = 3(x+4)^2 - 1$					
b)	$y = -\frac{1}{4}(x-6)^2 + 2$					
c)	$y = -(x-7)^2$					
d)	$y = \frac{1}{2}x^2 + 9$					

13. Sketch each parabola by the step pattern. Label the vertex.

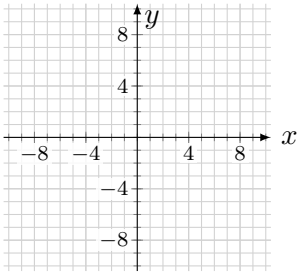
a) $y = (x+3)^2 - 8$



b) $y = -\frac{1}{2}(x-2)^2 + 5$



c) $y = 2(x+1)^2 - 6$



14. For a) and b), write the equation in vertex form. For c) and d), say in words what each number does to the graph of $y = x^2$. Work on lined paper.

a) $y = x^2$ reflected in the x-axis, compressed vertically by a factor of $\frac{1}{3}$, then moved 6 units left and 4 units up

b) the same shape as $y = x^2$, opening up, with its vertex at $(5,\ -2)$

c) $y = 4(x+2)^2 - 7$

d) $y = -3(x-1)^2 + 8$

15. Lin says that $y = -(x-4)^2 + 3$ has its vertex at $(-4,\ 3)$ and a minimum value of 3. Find both of Lin's errors. Give the correct vertex and say whether 3 is a maximum or a minimum. Work on lined paper.

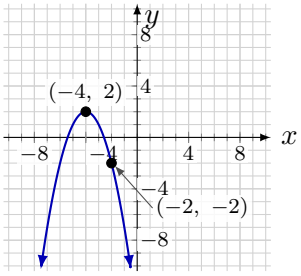
Mixed Review

These questions are in mixed order, as on a test. Ask what each one gives you and what it asks for. Work on lined paper unless a grid is printed.

16. A train pulls away from a station. The table shows how far it has gone, in metres, after each second. Complete the difference columns. Is the relation linear, quadratic, or neither? Explain.

x	y	First differences	Second differences
0	0.0		
1	1.7		
2	6.3		
3	14.5		
4	25.5		
5	40.1		

17.

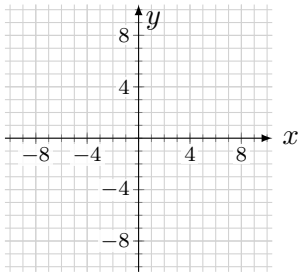


This parabola has the same shape as $y = x^2$. Write its equation in vertex form. Then check your equation with the marked point $(-2,\ -2)$.

$y =$ _____

18. You have this relation only in words: *the parent parabola reflected in the x-axis, then moved 1 unit right and 3 units up.* Write its equation. Complete the table, and use the differences to show that it is quadratic. Then sketch its graph.

x	y	First differences	Second differences
−2			
−1			
0			
1			
2			
3			
4			



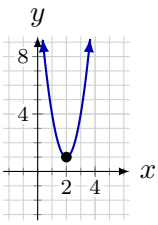
Equation:

_____

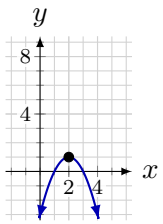
19. All four graphs have their vertex at $(2,\ 1)$. Match each graph with its equation. Then explain on lined paper how you told graph (ii) from graph (iv).

(i) $y = (x-2)^2+1$ (ii) $y = 3(x-2)^2+1$ (iii) $y = -(x-2)^2+1$ (iv) $y = \frac{1}{3}(x-2)^2+1$

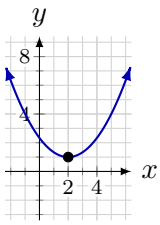
A



B



C



D

