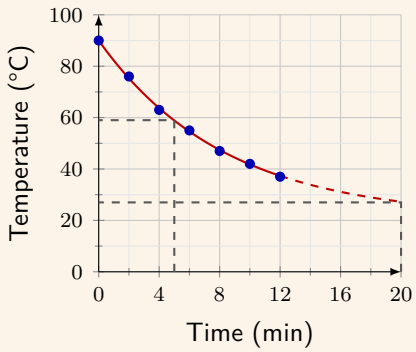
# Reviewing Quadratic Relations

Read each worked example, then try the questions that follow it. Plot and sketch on the grids printed here; work the rest on lined paper.

## Relations and Models

I poured a mug of hot chocolate and took its temperature every 2 minutes. I controlled the times, so **time is the independent variable**: first in each ordered pair, on the horizontal axis. Temperature depends on time: it is the **dependent variable**, on the vertical axis.

| Time (min) | Temp. (°C) |
|---|---|
| 0 | 90 |
| 2 | 76 |
| 4 | 63 |
| 6 | 55 |
| 8 | 47 |
| 10 | 42 |
| 12 | 37 |




**Describe the scatter plot in words**

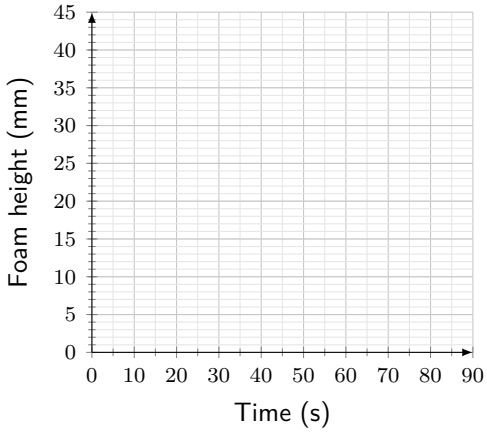
- **Direction:** negative. As time increases, the temperature decreases.
- **Strength:** strong. The points lie close to a smooth curve.
- **Shape:** non-linear. It falls quickly, then levels off.

A **curve of best fit** follows the trend, with about as many points above it as below.

**Predict from the curve.** At 5 min the curve gives about 59 °C. Five minutes lies inside the data (0 to 12 min): an **interpolation**, a prediction between known values. At 20 min it gives about 27 °C. Twenty minutes lies beyond the data: an **extrapolation**, a prediction outside the known values. I trust the interpolation more: an extrapolation assumes the trend carries on, and the drink cannot cool below room temperature.

**1.** A glass of root beer was poured, and the height of the foam on top was measured every 10 seconds.

| Time (s) | Foam (mm) |
|---|---|
| 0 | 40 |
| 10 | 29 |
| 20 | 20 |
| 30 | 16 |
| 40 | 13 |
| 50 | 11 |
| 60 | 10 |




a) Which variable is independent, and which is dependent?

b) Plot the data on the grid. Draw a curve of best fit, and extend it to 90 s.

c) Describe the relation in words: its direction, strength, and shape.

d) Use your curve to estimate the foam height at 25 s and at 90 s. Say which estimate is an interpolation and which is an extrapolation.

e) Which estimate do you trust more? Why?

**2.** I weighed a puppy once a week, from week 2 to week 12 after it was born, and drew a curve of best fit. Is each prediction an interpolation or an extrapolation?

a) its mass at week 7 b) its mass at week 20 c) its mass at birth, week 0

d) Which of the three predictions do you trust least? Explain.

## Finite Differences: Linear, Quadratic, or Neither?

In a table whose $x$ values are **evenly spaced**, subtract each $y$ value from the next to get the **first differences**. Subtract each first difference from the next to get the **second differences**.

**Linear:** the first differences are constant.
**Quadratic:** the first differences are not constant, but the second differences are.
**Neither:** neither column is constant.

For $y = 2x^2 - 3x - 1$ (right), the second differences are all 4: it is quadratic. **Measured data** give differences that are only roughly constant.

| $x$ | $y$ | First differences | Second differences |
|---|---|---|---|
| $-1$ | 4 | | |
| | | $-5$ | |
| 0 | $-1$ | | 4 |
| | | $-1$ | |
| 1 | $-2$ | | 4 |
| | | 3 | |
| 2 | 1 | | 4 |
| | | 7 | |
| 3 | 8 | | |

### Check the $x$ column first

The rule needs evenly spaced $x$ values. For $y = x^2$ with the rows $x = 0, 1, 3, 4$, the jump from 1 to 3 gives first differences 1, 8, 7, and the relation looks like "neither". Put in the missing row, $(2, 4)$, and the first differences become 1, 3, 5, 7, with second differences all 2.

**Which is faster?** Look at the first differences before you work out any second differences. If they are constant, stop: the relation is linear. If they grow by doubling or tripling, it is neither.

**3.** Complete the difference columns for each table. Then decide whether the relation is linear, quadratic, or neither, and say how you know. In c), a cart rolled $d$ metres in $x$ seconds.

**a)**

| $x$ | $y$ | First differences | Second differences |
|---|---|---|---|
| 1 | 5 | | |
| 2 | 1 | $1 - 5 = -4$ | $-4 - (-4) = 0$ |
| 3 | $-3$ | $-3 - 1 = -4$ | _____ |
| 4 | $-7$ | _____ | _____ |
| 5 | $-11$ | _____ | |

Each difference is the lower value minus the one above it.

**b)**

| $x$ | $y$ | First differences | Second differences |
|---|---|---|---|
| 0 | 1 | | |
| 1 | 3 | | |
| 2 | 9 | | |
| 3 | 27 | | |
| 4 | 81 | | |

_____

**c)**

| $x$ | $d$ | First differences | Second differences |
|---|---|---|---|
| 0 | 0.0 | | |
| 1 | 1.2 | | |
| 2 | 4.9 | | |
| 3 | 10.7 | | |
| 4 | 19.1 | | |

_____

**4.** Kai made this table and said, "The first differences are 9, 15, 48, 33. Neither column will be constant, so the relation is neither linear nor quadratic." Work on lined paper.

| $x$ | $y$ |
|---|---|
| 1 | $-1$ |
| 2 | 8 |
| 3 | 23 |
| 5 | 71 |
| 6 | 104 |

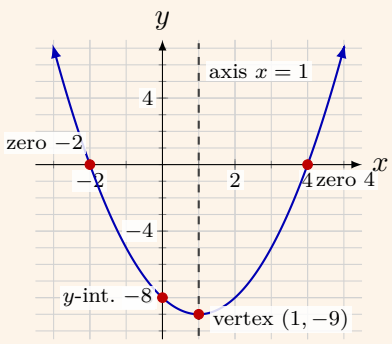
a) What did Kai miss?

b) The relation is quadratic. Continue the pattern in the first differences to find $y$ when $x = 4$. Then show that the second differences of the completed table are constant.

## Key Features of a Parabola, and Four Ways to Describe It

The turning point of a parabola is its **vertex**. The vertical line through it is the **axis of symmetry**: the two halves are mirror images in it. The $y$-coordinate of the vertex is the **minimum value** when the parabola opens up, the **maximum value** when it opens down. The **zeros** are the $x$-intercepts, the values of $x$ that make $y = 0$. The **$y$-intercept** is the value of $y$ when $x = 0$.

**One relation, four ways:** as a graph, as a table of values, in words, and as an equation.



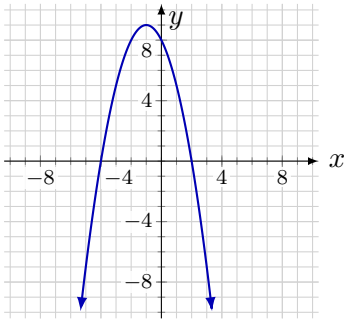

| $x$ | $y$ |
|---|---|
| $-2$ | 0 |
| $-1$ | $-5$ |
| 0 | $-8$ |
| 1 | $-9$ |
| 2 | $-8$ |
| 3 | $-5$ |
| 4 | 0 |

**In words.** A parabola that opens up, with its vertex at $(1, -9)$, its axis of symmetry $x = 1$, its minimum value $-9$, its zeros $-2$ and 4, and its $y$-intercept $-8$. $x$ may take any value; $y$ may take any value with $y \geq -9$.

**As an equation.** $y = x^2 - 2x - 8$. At $x = 0$: $y = -8$.

**Where to look in a table.** The zeros are the rows where $y = 0$. Equal $y$ values pair up around the axis, so the vertex is the row in the middle of the pairs.

**5.** You have this relation only as a graph. Describe its key features.




Opens _____ Zeros _____

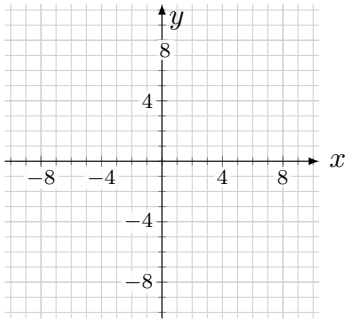
Vertex _____ $y$-intercept _____

Axis _____ $x$ may take _____

Max or min value _____ $y$ may take _____

**6.** You have this relation only as an equation: $y = x^2 - 6x + 5$. Describe it the other three ways: as a table of values, as a graph, and in words.

| $x$ | $y$ |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |




**In words:**

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_____

_____

_____

## Transformations and the Step Pattern

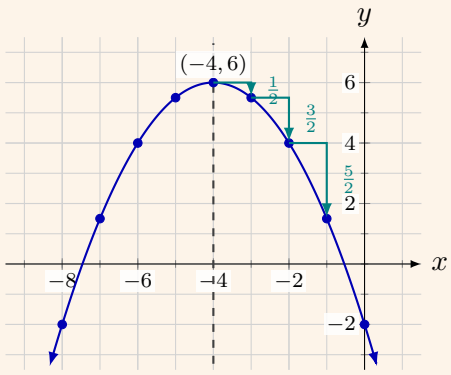
Every parabola in vertex form, $y = a(x - h)^2 + k$, is the parent parabola $y = x^2$ moved or reshaped.

- $k$ slides the graph up ($k > 0$) or down ($k < 0$): a **vertical translation**.
- $h$ slides it right ($h > 0$) or left ($h < 0$): a **horizontal translation**. Watch the sign: $y = (x + 3)^2$ is $(x - (-3))^2$, so $h = -3$, 3 units left.
- $a$ multiplies every $y$ value. When $a < 0$ the graph is **reflected in the $x$-axis** and opens down. When $a$ is more than 1 in size it is a **vertical stretch** (narrower); less than 1 in size, a **vertical compression** (wider).

So the vertex is $(h, k)$, the axis of symmetry is $x = h$, and $k$ is the maximum or minimum value.

**Sketch $y = -\frac{1}{2}(x + 4)^2 + 6$ by the step pattern**

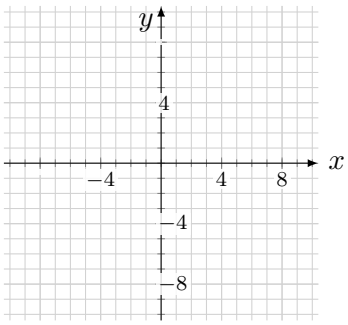
1. Plot the vertex, $(-4, 6)$.
2. For $y = x^2$, the steps from the vertex are over 1 and up 1, 3, 5, 7. Multiply each by $a = -\frac{1}{2}$: down $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$. That gives $(-3, 5.5)$, $(-2, 4)$, $(-1, 1.5)$, $(0, -2)$.
3. Mirror the points in the axis $x = -4$: $(-5, 5.5)$, $(-6, 4)$, $(-7, 1.5)$, $(-8, -2)$.
4. Join them with a smooth curve. Check one: at $x = 0$, $-\frac{1}{2}(0 + 4)^2 + 6 = -2$.



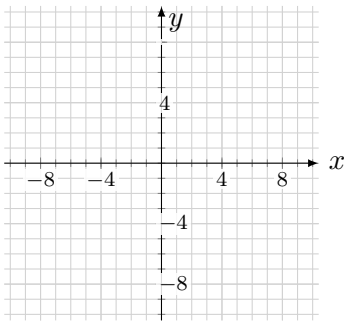

**Which is faster?** From vertex form, the step pattern: no table, and the vertex comes first. Make a table of values only when the equation is not in vertex form. On a $-10$ to 10 grid, stop when the next step would leave the grid.

**7.** Sketch each parabola by the step pattern. Label the vertex.

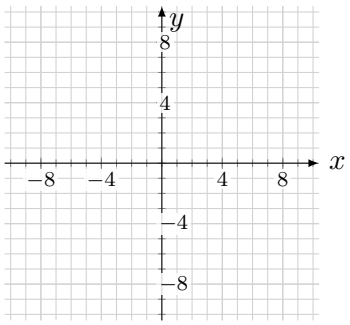
a) $y = \frac{1}{2}(x + 5)^2 - 4$




b) $y = -3x^2 + 8$




c) $y = -(x - 3)^2 + 6$




**8.** For a) and b), say in words what each number does to the graph of $y = x^2$, and give the vertex and the axis of symmetry. For c) and d), write the equation in vertex form. Work on lined paper.

a) $y = 4(x + 6)^2 - 1$

b) $y = -\frac{1}{3}(x - 5)^2 + 2$

c) $y = x^2$ reflected in the $x$-axis, stretched vertically by a factor of 3, then moved 2 units left and 5 units up

d) $y = x^2$ compressed vertically by a factor of $\frac{1}{4}$, then moved 7 units right

## Write the Equation from a Graph, Two Ways

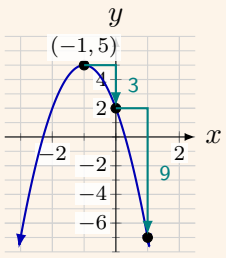
Read the vertex $(h, k)$ from the graph and put it into $y = a(x - h)^2 + k$. Only $a$ is left to find.

### By the step pattern

The vertex is $(-1, 5)$, so $y = a(x + 1)^2 + 5$.

From the vertex, 1 across goes down 3, so $a = -3$. The next step goes down 9, which is $3a$: it checks.

$y = -3(x + 1)^2 + 5$




### By substitution

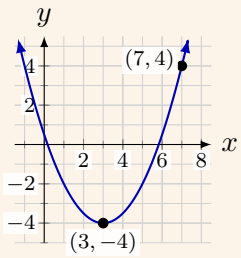
The vertex is $(3, -4)$, so $y = a(x - 3)^2 - 4$.

Substitute the point $(7, 4)$:

$4 = a(7 - 3)^2 - 4$

$8 = 16a$, so $a = \frac{1}{2}$.

$y = \frac{1}{2}(x - 3)^2 - 4$



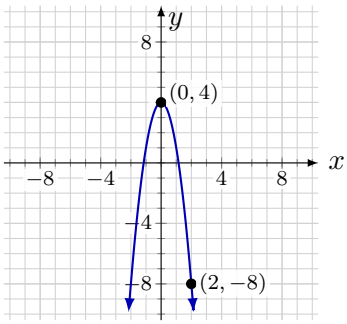

Check with another point. For $y = \frac{1}{2}(x - 3)^2 - 4$: $(-1, 4)$ gives $\frac{1}{2}(-1 - 3)^2 - 4 = 8 - 4 = 4$. ✓

### Which is faster?

- Use the step pattern when the points one and two steps from the vertex sit on grid corners.
- Use substitution when they do not: here, one step from $(3, -4)$ lands at $(4, -3.5)$, between grid lines. Substitution works with any point you can read exactly, however far it is from the vertex.
- A parabola that opens down has a negative $a$: the steps go down.

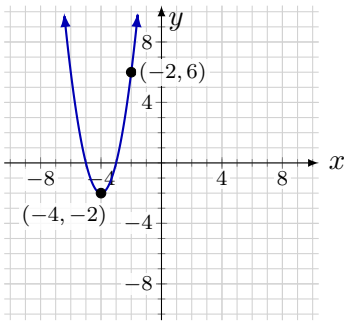
**9.** Write the equation of each parabola in vertex form. The vertex and one more point are marked.

a)



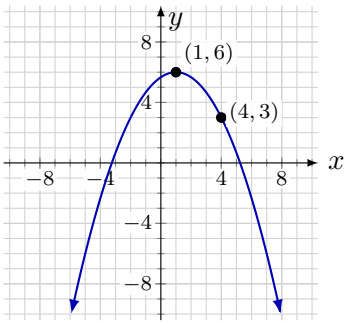

$y =$ _____

b)




$y =$ _____

c)




$y =$ _____

**10.** Write each equation in vertex form. Work on lined paper.

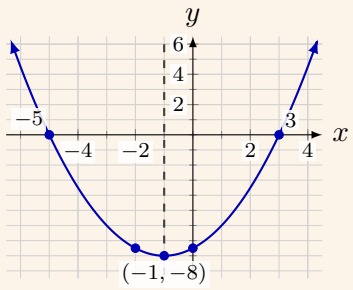
a) The vertex is $(-6, 1)$, and the parabola passes through $(-2, 9)$.

b) Same shape as $y = 4x^2$, but it opens down; its vertex is $(3, -2)$.

c) The vertex is $(2, -3)$, and the $y$-intercept is 9.

## Intercepts Form: $y = a(x - r)(x - s)$

At $x = r$ or $x = s$ a bracket is 0, so $y = 0$: the **zeros** are $r$ and $s$, each with the opposite sign to the number in its bracket. The axis of symmetry is midway, at $x = \frac{r+s}{2}$, and the vertex is on it.




**Sketch** $y = \frac{1}{2}(x + 5)(x - 3)$

Zeros $-5$ and 3. Axis midway: $x = \frac{-5+3}{2} = -1$. Vertex: $\frac{1}{2}(-1 + 5)(-1 - 3) = \frac{1}{2}(4)(-4) = -8$, so $(-1, -8)$. $y$-intercept: $\frac{1}{2}(0 + 5)(0 - 3) = -7.5$, with its partner $(-2, -7.5)$ across the axis. $a = \frac{1}{2} > 0$, so it opens up.

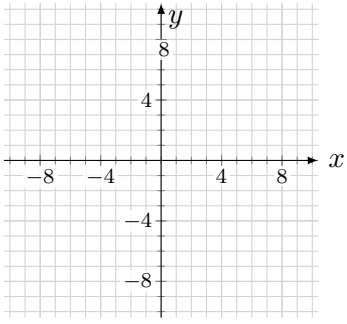
**Write the equation from the zeros and one more point**

Zeros $-2$ and 6, through $(4, 6)$. Put the zeros in the brackets with the opposite sign: $y = a(x + 2)(x - 6)$. Substitute the point, which must not be a zero: $6 = a(4 + 2)(4 - 6) = -12a$, so $a = -\frac{1}{2}$ and $y = -\frac{1}{2}(x + 2)(x - 6)$.

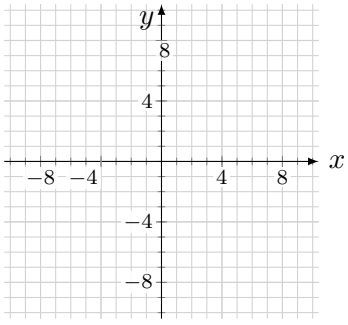
**Which form is faster?** Sketch from intercepts form when you know the zeros, from vertex form when you know the vertex.

**11.** Sketch each parabola. Label the zeros, the vertex, and the $y$-intercept, and draw the axis of symmetry.

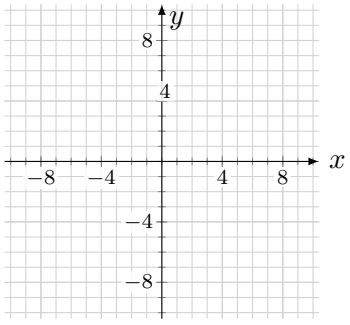
a) $y = (x - 1)(x - 5)$




b) $y = 2(x + 4)(x + 1)$

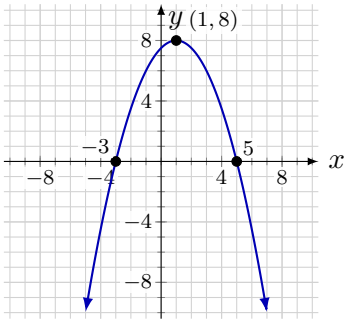



c) $y = -\frac{1}{2}(x - 1)(x + 7)$

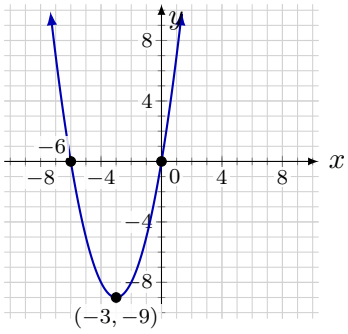



**12.** Write the equation of each parabola in intercepts form. The labelled points are exact. Work on lined paper.

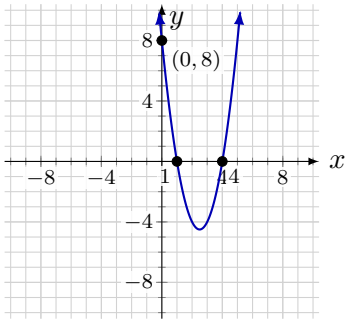
a)




b)




c)




**13.** Which of these can be written in intercepts form? For each one that can, write it. Explain each one that cannot. Work on lined paper.

a) $y = (x - 4)^2$ b) $y = x^2 + 3$ c) $y = -(x + 2)^2 + 1$

## Modelling with Vertex Form or Intercepts Form

Choose the form from what you know. Knowing the highest or lowest point points to **vertex form**; knowing where something meets the ground points to **intercepts form**. One more point gives $a$. Then say what $x$ and $y$ stand for, with units.

### A pass, in vertex form

A basketball pass leaves a player's hands 2 m above the floor and reaches its greatest height, 4.5 m, when it is 2.5 m away across the floor.

The greatest height is the vertex, $(2.5, 4.5)$: $y = a(x - 2.5)^2 + 4.5$. The release point $(0, 2)$ gives $2 = a(0 - 2.5)^2 + 4.5$, so $6.25a = -2.5$ and $a = -0.4$.

$$y = -0.4(x - 2.5)^2 + 4.5$$

$x$ is the distance across the floor and $y$ the height, both in metres. At 4 m across: $-0.4(1.5)^2 + 4.5 = 3.6$ m.

### A hop, in intercepts form

A kangaroo leaves the ground, reaches a greatest height of 1.8 m, and lands 6 m from where it took off.

Put the take-off at the origin. The zeros are 0 and 6: $y = a(x - 0)(x - 6)$. The top is midway, at $(3, 1.8)$: $1.8 = a(3)(3 - 6) = -9a$, so $a = -0.2$.

$$y = -0.2x(x - 6)$$

$x$ is the distance from the take-off and $y$ the height above the ground, both in metres. At 1 m: $-0.2(1)(-5) = 1$ m.

### Interpolation and extrapolation with a model

A model is trustworthy only where the situation it describes is real. Using it between known values is interpolation; using it beyond them is extrapolation, and less certain. The kangaroo model gives $-0.2(8)(2) = -3.2$ m at 8 m: the kangaroo has landed by then, and the model describes only $0 \le x \le 6$.

**14.** The cross-section of a skateboard half-pipe is a parabola. Its lowest point, in the middle, is 0.2 m above the ground. The left edge is 3.4 m above the ground and 4 m across from the middle. Let $x$ be the distance across from the left edge and $y$ the height, both in metres. Work on lined paper.

a) Write a model in vertex form.

b) How wide is the half-pipe? Explain without calculating.

c) How high is the surface 1 m in from the left edge? Where else is it that high?

d) What do $h$ and $k$ tell you about the half-pipe?

**15.** A golf ball is hit from the ground and lands 50 m away. Its greatest height is 15 m. Let $x$ be the distance from where it was hit and $y$ the height, both in metres. Work on lined paper.

a) Write a model in intercepts form.

b) How high is the ball 10 m from where it was hit? Where else is it that high?

c) A hedge 2.4 m tall stands 6 m from where the ball was hit. Does the ball clear it? Explain.

d) What does the model give at 55 m? Why does it make no sense?

**16.** A food truck tried prices from \$4 to \$12 for its lunch special. Its weekly profit, $P$ hundred dollars, at a price of $p$ dollars fits the model $P = -2(p - 9)^2 + 50$. Work on lined paper.

a) Predict the profit at a price of \$6. Is this an interpolation or an extrapolation?

b) Predict the profit at a price of \$15. Is this an interpolation or an extrapolation?

c) Which price gives the greatest profit? What is that profit?

d) Which prediction, a) or b), do you trust more? Why?

## Negative and Zero Exponents

| Power | Value |
|---|---|
| $6^3$ | 216 |
| $6^2$ | 36 |
| $6^1$ | 6 |
| $6^0$ | 1 |
| $6^{-1}$ | $\frac{1}{6}$ |
| $6^{-2}$ | $\frac{1}{36}$ |

### The pattern route

Each step down the table lowers the exponent by 1 and divides the value by 6. Nothing stops the pattern at $6^1$: one more step gives $6 \div 6 = 1$, so $6^0 = 1$. Then $\frac{1}{6}$, $\frac{1}{36}$, and so on.

### The law route

The law $a^m \div a^n = a^{m-n}$ must agree with arithmetic.

$6^2 \div 6^2 = 6^0$, and $36 \div 36 = 1$.

$6^2 \div 6^4 = 6^{-2}$, and $\dfrac{36}{1296} = \dfrac{1}{36}$.

For any base $a$ that is not 0, and any whole number $n$:

$$a^0 = 1 \qquad a^{-n} = \frac{1}{a^n}$$

$$9^{-2} = \frac{1}{9^2} = \frac{1}{81} \qquad \left(\tfrac{2}{3}\right)^{-2} = \left(\tfrac{3}{2}\right)^2 = \tfrac{9}{4}$$

$$(-2)^{-2} = \frac{1}{(-2)^2} = \frac{1}{4} \qquad -2^{-2} = -\frac{1}{2^2} = -\frac{1}{4}$$

**A negative exponent does not make the answer negative.** It means "one over": divide by 9 twice, starting at 1, so $9^{-2} = \frac{1}{81}$, which is positive.

**Watch the brackets.** In $(-2)^{-2}$ the base is $-2$; in $-2^{-2}$ the base is 2, and the negative sign is applied last.

**Which is faster?** For a fraction raised to a negative exponent, flip the fraction and make the exponent positive: $\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2$.

**17.** Evaluate. Give each answer as a whole number or a fraction.

a) $12^0$ b) $4^{-3}$ c) $(-2)^{-3}$ d) $-6^0$

e) $\left(\frac{1}{3}\right)^{-2}$ f) $\left(\frac{2}{5}\right)^{-1}$ g) $10^{-4}$ h) $2^{-1} + 2^0 + 2^1$

**18.** The area covered by algae on a pond triples each week. The area, $A$ square metres, $t$ weeks after June 1 is $A = 810 \times 3^t$. Work on lined paper.

a) What does $t = 0$ give? What does it stand for?

b) What does $t = -3$ stand for? Find $A$ then.

c) How many weeks before June 1 did the algae cover $10\,\text{m}^2$?

d) Will the area ever be 0? Explain.

**19.** Work on lined paper.

a) Ravi wrote $5^{-3} = -15$. Explain both of his errors, and give the correct value.

b) Explain why $4^0 = 1$, using a pattern or the exponent law for division.

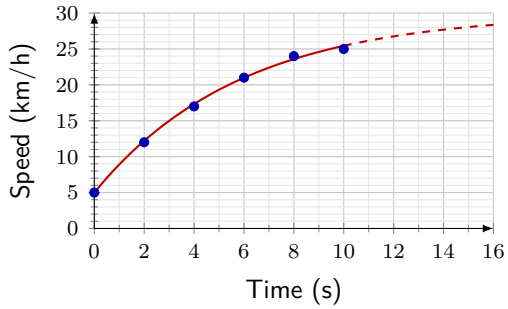
## Mixed Review: Knowing Where to Start

On a test, nothing tells you which page of this review a question comes from. When you are not sure how to begin:

- What does the question give you: words, a table of values, an equation, or a graph? What does it ask for?
- Given a table, check the $x$ column first. Given the vertex, use vertex form; given the zeros, intercepts form.
- Write down what you know: label the vertex or the zeros, define $x$ and $y$ with units, or make a quick sketch.

These questions are in mixed order. Decide for yourself how to approach each one. Work on lined paper unless a grid is printed.

**20.** A cyclist starts down a hill. I recorded her speed every 2 s for 10 s and drew a curve of best fit.




a) Describe the relation in words: its direction, strength, and shape.

b) Use the curve to estimate her speed at 7 s and at 14 s.

c) Which estimate is an interpolation, and which is an extrapolation? Which do you trust more? Why?

**21.** A frisbee is thrown from 1.2 m above the ground. It reaches its greatest height, 3.2 m, when it is 10 m from the thrower. Let $x$ be the distance from the thrower and $y$ the height, both in metres.

a) Write a model in vertex form.

b) How high is it 18 m out?

c) Where else is it that high?

**22.** Is this relation linear, quadratic, or neither? Find the first and second differences, and say how you know.

| $x$ | $y$ |
|---|---|
| 0 | 30 |
| 2 | 26 |
| 4 | 18 |
| 6 | 6 |
| 8 | −10 |

**23.** Evaluate. Give each answer as a whole number or a fraction.

a) $2^{-5}$

b) $(-5)^0$

c) $\left(\frac{1}{4}\right)^{-3}$

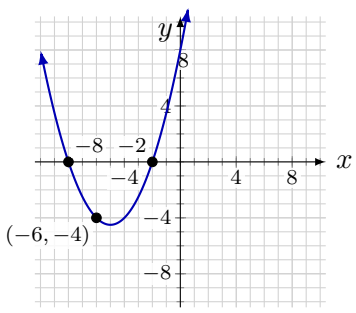
d) $8^{-1} + 8^0$

**24.** Jun says the vertex of $y = (x+3)^2 - 5$ is $(3, -5)$. Explain Jun's error, and give the correct vertex.

**25.** A parabola has zeros 2 and 8, and its $y$-intercept is $-16$. Find its equation and its vertex.

**26.** Which is greater, $8^{-2}$ or $2^{-8}$? Explain without a calculator.

**27.**




Write the equation of this parabola in intercepts form. The labelled points are exact.

$y =$ _____

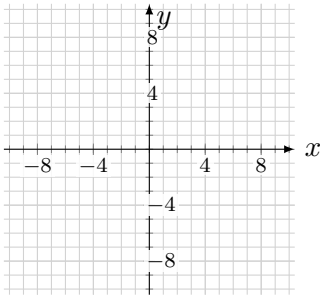
**28.**

| $x$ | $y$ |
|---|---|
| 0 | 4 |
| 1 | 7 |
| 2 | 10 |
| 3 | 13 |
| 4 | 16 |

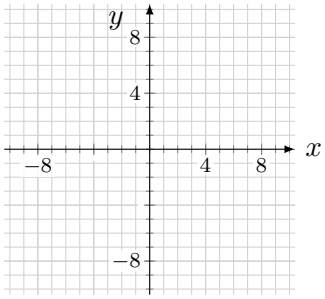
Mia made this table and said, "The first differences are 3, 3, 3, 3, so the second differences are 0, 0, 0. They are constant, so the relation is quadratic." Explain Mia's error. What kind of relation is it?

**29.** Sketch each parabola by the step pattern. Label the vertex and the $y$-intercept.

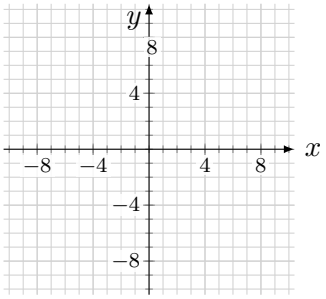
a) $y = -\frac{1}{2}(x-4)^2 + 5$




b) $y = 3(x+2)^2 - 7$

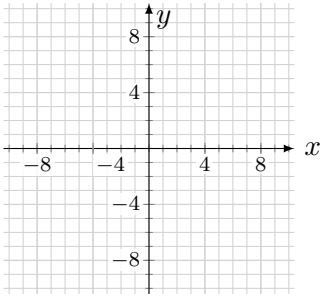



c) $y = \frac{1}{4}(x+6)^2$




**30.** You have this relation only in words: *the parent parabola reflected in the $x$-axis, then moved 2 units left and 4 units up.* Describe it the other three ways: as an equation, as a table of values, and as a graph. Then write it in intercepts form.

| $x$ | $y$ |
|---|---|
| −5 | |
| −4 | |
| −3 | |
| −2 | |
| −1 | |
| 0 | |
| 1 | |




**Equation:**

_____

**Intercepts form:**

_____

**31.** A jet of water from a lawn sprinkler on the ground lands 10 m away; 2 m from the sprinkler it is 2.4 m high. With $x$ and $y$ in metres, write a model in intercepts form, and find the greatest height of the water.

**32.** A parabola has the same shape as $y = \frac{1}{2}x^2$ but opens down, has axis of symmetry $x = 3$, and passes through $(5, 4)$. Write its equation in vertex form.

**33.** Without making a table, give the second differences of $y = -3(x+1)^2 + 2$ when $x$ steps by 1. Explain how you know.